# RESEARCH STATEMENT

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## 1. Introduction

My research focus is stochastic analysis. I study both stochastic differential equations and stochastic partial differential equations. My current project with my advisor Cheng Ouyang and collaborator Le Chen is investigating the stochastic heat equation on manifolds. We constructed a family of noise on manifolds, which we call colored noise on manifolds. Past projects include investigating the self-intersections of rough differential equations driven by fractional Brownian motion (fBm). The methods I use come from the theory of rough paths, differential geometry, the Itô calculus, and the Malliavin calculus.

I believe it is important to approach mathematical research from an interdisciplinary perspective, and I am looking to improve my knowledge of algebra and geometry to be able to apply tools from these areas to my research in analysis. Another goal is to communicate ideas from probability and stochastic analysis to mathematicians from other fields as well as scientists and researchers outside of mathematics. I hope to use tools such as computers to design mathematical illustrations and experiments as well.

## 2. Rough Paths and Fractional Brownian Motion

**Results.** Our results in [7] on rough paths concern the self-instersection of the solution to a rough differential equation driven by fractional Brownian motion. Pólya, Erdös, Lévy, and others studied questions related to whether and how many self-intersections occur for Brownian motion in [39] [11] [12] [13] [28] [29]. We proved in [7] that if $Z_t$ is the solution to a rough differential equation of the form

$$Z_t = x_0 + \int_0^t V_0(Z_s)ds + \sum_{i=1}^{d} \int_0^t V_i(Z_s)dX_s^i, \tag{2.1}$$

where $X_t$ is an fBm and $V^i, 0 \leq i \leq d$, is a vector field satisfying elliptic conditions then the following theorem holds:

**Theorem 2.1.** *Let $H > \frac{1}{4}$, and let $Z_t$ denote the solution to a $d$-dimensional stochastic differential equation of the form* (2.1)*, then*

$$\mathit{Cap}_{(r,q)}\left(\{Z_t = Z_s \,\mathit{for}\; 0 \leq s < t \leq T\}\right) = 0$$

*if*

$$\frac{2}{H} + rq < d.$$

$\text{Cap}_{(r,q)}$ denotes the $(r,q)$-capacity, an outer measure finer than the probability measure on the state space $\Omega$. If the $(r,q)$-capacity of a set is zero then the probability measure of that set is zero, but not vice-versa. The $(0,1)$-capacity is equal to the probability measure. Takeda proved in [41] that self-intersecting paths of Brownian motion have $(r,q)$-capacity zero given $4+rq<d$, corresponding to the $H=1/2$ case. In 2018 Li and Qian proved that self-intersecting paths of fBm with Hurst parameter $H$ form a set of zero $(r,q)$-capacity when $d>rq+2/H$ [30]. We extend this result to include all rough differential equations driven by fBm, and we apply the rough path theory in a manner similar to Boedihardjo, Geng, Liu, and Qian's 2016 paper on the signature of Brownian motion [2]. The rough path machinery allows one to apply the straightforward methods of Kakutani's 1944 proof in [21].

**Theory of Rough Paths and Rough Differential Equations.** We review some of the background from Terry Lyons' theory of rough paths ([33] is a great introduction) to motivate the results. A rough path $\mathbf{X}_{s,t}$ lives in the tensor algebra

$$T(V)=\bigoplus_{n=0}^{\infty}(V)^{\otimes n},$$

where $V^{\otimes 0}=F$, the underlying field of $V$. $\mathbf{X}_{s,t}$ is a path which satisfies the following multiplicative identity

$$\mathbf{X}_{s,t}=\mathbf{X}_{s,u}\otimes\mathbf{X}_{u,t},$$

for $s<u<t$ [33]. We write the rough path in coordinates as follows

$$\mathbf{X}_{s,t}=\left(X^0_{s,t},X^1_{s,t},\ldots,X^i_{s,t},\ldots\right),$$

where $X^i_{s,t}\in V^{\otimes i}$. The motivation for the multiplicative condition is Chen's identity for the signature $S$ of a path $\gamma$, which also lives in the tensor algebra,

$$S(\gamma)=\left(1,\int_{0\le s_0\le t}d\gamma(s_0),\int_{0\le s_0\le s_1\le t}d\gamma(s_0)\otimes d\gamma(s_1),\ldots\right).$$

For paths $\gamma_1,\gamma_2$, the signature $S$ commutes with concatenation in the following sense $S(\gamma_1*\gamma_2)=S(\gamma_1)\otimes S(\gamma_2)$, where $*$ denotes concatenation and $\otimes$ denotes the tensor product on the graded tensor algebra. The multiplicative condition means that concatenating successive increments of a rough path is the same as taking their tensor product, just as though it were the signature of a path.

**Fractional Brownian Motion.** fBm was first introduced by Kolmogorov in [26], but the name was not coined until Mandelbrot and Van Ness wrote [35]. An fBm is a mean zero Gaussian process, whose properties depend on the Hurst parameter $H$. For $H=1/2$, the fBm is a Brownian motion with independent increments. If $H<1/2$ the increments are negatively correlated, and if $H>1/2$ they are positively correlated. For $H\le 1/4$ the path becomes a space filling curve, which is not amenable to our methods.

In [9] Coutin and Qian showed that for $1/4<H<1$ the dyadic interpolations

$$X^{(n)}_t=X_{\frac{k}{2^n}}+\left(t-\frac{k}{2^n}\right)\cdot 2^{-n}\cdot\left(X_{\frac{k+1}{2^n}}-X_{\frac{k}{2^n}}\right),$$

defined for $t \in [\frac{k}{2^n}, \frac{k+1}{2^n}) \subset [0, T]$, converge in $p$-variation for $p > \frac{1}{H}$. It is helpful to visualize the convergence in the following diagram,

$$\begin{CD} X_t^{(n)} @>>> X_t \\ @VVV @VVV \\ S(X_t^{(n)}) @>>> \mathbf{X}_{0,t} \end{CD}$$

This means that for $H > 1/4$, the canonical lift for the fBm is a well defined geometric rough path. Because the Itô-Lyons solution map $\Psi$ is continuous in $p$-variation we can illustrate the convergence of the solution $Z_t$ in another diagram,

$$\begin{CD} X_t^{(n)} @>>> X_t \\ @V{\Psi}VV @V{\Psi}VV \\ Z_t^{(n)} @>>> Z_t \end{CD}$$

Note $Z_t^{(n)}$ is an approximation to the solution which also converges in $p$-variation becuase $\Psi$ is continuous in this norm. Convergence also preserves properties related to the Malliavin calclulus, such as quasi-continuity, which we will describe in the next section. These diagrams are inspired by Milnor's monograph on differential topology [37]. The idea is to illustrate the relationships between the target and image spaces and to compare properties before and after the limit is taken. I hope to use this perspective to illustrate and solve mathematical questions in an interdisciplinary context during my career.

**Malliavin Calculus.** The key idea of the Malliavin calculus involves associating the sample space and probability measure $(\Omega, \mathcal{F}, \mathbb{P})$ with a Hilbert space $\mathcal{H}$ [38]. Gaussian random variables are associated to elements $h, g \in \mathcal{H}$ by a map $W(\cdot)$ that preserves the Hilbert norm,

$$\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_{\mathcal{H}}.$$

These random variables form an isonormal Gaussian family. We call random variables $F : \Omega \to V$ of the following form

$$F = f(W(h_1), \ldots, W(h_n))$$

cylindrical functions, where $f \in C_p^\infty(V^n)$, the class of smooth functions where all partial derivatives have sub-polynomial growth. We then define the Malliavin derivative

$$\mathbb{D}F = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n)) h_i.$$

In the simplest case, where $F = W(h)$, this associates the random variable with a direction in $\mathcal{H}$, namely the subspace spanned by $h$. More complex random variables are built out of simpler components, and we associate them with direction and change as well, just like in calculus. We can take higher derivatives $\mathbb{D}^k$ in an analogous manner, and we define a Sobolev norm

$$||F||_{r,q} = (\mathbb{E}[|F|^q] + \mathbb{E}[|\mathbb{D}F|] + \cdots + \mathbb{E}[|\mathbb{D}^r F|^q])^{1/q}.$$

We can complete the space of cylindrical functions with respect to this norm to obtain the $(r, q)$-Sobolev space $\mathbb{D}_{r,q}$. One definition for the capacity of a set is the infimum for the Sobolev norm of functions in $\mathbb{D}_{r,q}$ approximating the indicator function. Other definitions involve the Ornstein-Uhlenbeck semi-group or the classical potential theory, but they are all mutually absolutely continuous [20][38][40]. We used tools from [34] to show that the solution map and canonical lift map preserve the quasi-continuity property. This allowed us to apply the Chebyshev inequality for capacity, which says that for a random variable $F$

$$\text{Cap}_{r,q}(|F| > R) \leq \frac{M_{r,q}||F||_{r,q}}{R}.$$

Using this machinery from Malliavin calculus in the background, our proof in [7] involves bounding the size of dyadic increments above and below. The key fact is that if $Z_t = Z_s$ for distinct $t \neq s$ with $s \in [s_0, s_1]$ and $t \in [t_0, t_1]$, then $|Z_{t_0} - Z_{s_0}| \leq |Z_t - Z_{t_0}| + |Z_s - Z_{s_0}|$. We bound the capacity of the set of paths with maximum displacement on a fixed interval greater than a constant $\eta$. We also bound the capacity of the set of paths with total path displacement less than $\eta$. In this way we control the capacity of the set of paths with large oscillations. We then use the sub-additivity of capacity to apply these bounds to shrinking dyadic intervals of $[s_0, s_1]$ and $[t_0, t_1]$. Sending $\eta$ to zero, we show that self-intersections have capacity zero given our criteria. This is essentially the same technique as Kakutani used in 1944 with the rough path and Malliavin calculus machinery largely in the background. Putting the problem in the context of the rough path theory and Malliavin calculus allowed us to considerably generalize past results.

## 3. Colored Noise on Manifolds and Stochastic Heat Equation

**Results.** Our results so far are summarized in the following theorem:

**Theorem 3.1.** *If the noise $\dot{W}$ satisfies Dalang's condition, then the solution $u(t, x)$ to (3.2) exists and is unique. In addition the second moment of the solution satisifes the following exponential upper and lower bounds*

$$ce^{kt} \leq \mathbb{E}[|u(t, x)|^2] \leq Ce^{Kt},$$

*where $k, K$ are fixed constants derived from controls on the self-iterated $\mathcal{L}_n$ operator defined below. The solution $u(t, x)$ is also $\beta_1$-Hölder continuous in time and $\beta_2$-Hölder continuous in space for $\beta_1 \in (0, \zeta/2)$ and $\beta_2 \in (0, \zeta)$ when there exists $\zeta > 0$ satisfying*

$$\sum_{k \in \mathbb{Z}^d, k \neq 0} \frac{\mathcal{F}(f)(k)}{|k|^{2(1-\zeta)}} < \infty,$$

*where $\mathcal{F}(f)(k)$ denotes the Fourier coefficient of the noise correlation $f$ corresponding to $k \in \mathbb{Z}^d$. Note that this condition involving $\zeta$ is stronger than Dalang's condition.*

**Colored Noise on Manifolds.** My research program in stochastic partial differential equations (spdes) is centered around our construction of a family of colored noise adapted to manifold geometry. The noise is constructed in the frequency domain. For many manifolds $M$, the eigenvectors of the Laplace-Beltrami operator $\Delta$ form a basis for $L^2(M)$. Given $\phi, \psi \in L^2(M)$, we can define $W(1_{[0,t]}\phi)$ and $W(1_{[0,s]}\psi)$ to be Gaussian variables with

$$\mathbb{E}\left[W(1_{[0,t]}\phi)W(1_{[0,s]}\psi)\right] = (s \wedge t)\langle \phi, \psi\rangle_{L^2(M)},$$

where the inner product on $L^2(M)$ can be evaluated in the frequency domain using the expansion in terms of eigenvalues. This defines a white noise on the manifold, which is too rough and will require regularity structures and other advanced machinery to integrate. To make the white noise more regular, we modify the inner product to soften the noise from white noise to colored noise in space and white noise in time.

Our current work focuses on studying spdes driven by our noise on the torus, which is a simplified or "toy" model we hope to generalize to an arbitrary manifold. On the torus $\mathbb{T}^d = [-\pi, \pi]$, the eigenvectors of $\Delta$ can be described explicitly: $\{e_k = e^{-ik\cdot x} : k \in \mathbb{Z}^d\}$. In this case $\phi, \psi \in L^2(\mathbb{T}^d)$ can be described using their fourier decomposition

$$\phi(x) = \sum_{k\in\mathbb{Z}^d} a_k e^{-ik\cdot x} \text{ and } \psi(x) = \sum_{k\in\mathbb{Z}^d} b_k e^{-ik\cdot x},$$

and we define $W(1_{[0,t]}\phi), W(1_{[0,s]}\psi)$ to be Gaussian variables with correlation

$$\mathbb{E}[W(1_{[0,t]}\phi)W(1_{[0,s]}\psi)] = (s \wedge t)\left(\varrho a_0 \bar{b}_0 + \sum_{k\in\mathbb{Z}^d, k\neq 0} \frac{a_k \bar{b}_k}{|k|^\alpha}\right),$$

where $\alpha$ and $\varrho$ are tuneable parameters. The noise can be introduced equivalently as $\dot{W}(t,x)$ with

(3.1)

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)G_\alpha(x-y) = \delta(t-s)\left(\varrho + (2\pi)^{-d/2}\sum_{k\in\mathbb{Z}^d, k\neq 0}\frac{e^{ik\cdot(x-y)}}{|k|^\alpha}\right).$$

This family of noise is intrinsic, and because it only depends on the spectral decomposition of the Laplacian it should be possible to generalize to an arbitrary manifold in a straightforward way.

**Stochastic Heat Equation.** The theory of spdes is based on the measure valued martingale and the Walsh integral [10]. These play the role of the martingale and the Itô integral in the theory of stochastic differential equations. The Itô-Burkholder isometry plays the role of the Itô isometry. Once the multi-variable integration theory is constructed, one seeks to show that the solution $u(t,x)$ satisfies an integral equation implied by the PDE expression.

Our research focuses the parabolic Anderson model (pAm) of the stochastic heat equation (she). The pAm is named for the physicist P.W. Anderson who created a model for spin diffusion and impurity band conduction in semiconductors in [1], where clustered phenemonon occur in a noisy environment. This is one of a number of physical phenomena where clusters appear [15], and the problem has been approached using discrete models in [27] as well as continuous models in [3] and [24]. They are related insofar as the density of a stochastic process in a random potential solves the spde for the pAm.

In a paper currently in progress with collaborators Cheng Ouyang and Le Chen we analyze the pAm on $\mathbb{T}^d$, which takes the form

$$u(t,x) = \int_{\mathbb{T}^d} \bar{G}(t,y,x)\mu(dy) + \int_0^t \int_{\mathbb{T}^d} \bar{G}(t-s,y,x)\lambda u(s,y)M(ds,dy), \tag{3.2}$$

where $\lambda$ is a constant, $\bar{G}$ is the heat kernel for the torus $\mathbb{T}^d$, and $M(ds,dy)$ is the noise. The heat kernel on the torus centered at $x_0$ is the sum of heat kernels on $\mathbb{R}^d$

$$\bar{G}(t,x_0,x) = \sum_{k\in\mathbb{Z}^d} G(t, x-x_0+k2\pi) = \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} e^{-\frac{|x-x_0+2k\pi|^2}{2t}}.$$

**The $\triangleright$ Operator and $\mathcal{L}_n$.** Le Chen developed an iterative method for analyzing the solution to the she on $\mathbb{R}^d$ in his PhD thesis [6] and applied it with collaborator Kunwoo Kim in [5]. If $f$ is the correlation function for the noise, we can evaluate the Itô-Burkholder isometry recursively

$$\begin{aligned}\mathbb{E}[u(t,x)u(t,x')] = &\int_{\mathbb{T}^d} \bar{G}(t,y,x)\mu(dy) \int_{\mathbb{T}^d} \bar{G}(t,y',x')\mu(dy') \\ &+ \lambda^2 \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathbb{E}[u(s,z)u(s,z')]\bar{G}(t-s,x,z)\bar{G}(t-s,x',z')f(z-z')dsdzdz'.\end{aligned}$$

In turn this suggests the $\triangleright$ operator, which we modified slightly from Le Chen's original definition for our context,

$$h \triangleright w(t,x_0,x,x_0',x') = \int_0^t ds \int\int_{\mathbb{T}^{2d}} dzdz' h(t-s,z,x,z',x',)w(s,x_0,z,x_0',z')f(z-z').$$

We can define $\mathcal{L}_n$ recursively using the $\triangleright$ operator as follows

$$\mathcal{L}_n(t,x_0,x,x_0',x') = \begin{cases} G(t,x_0,x)G(t,x_0',x') & \text{if } n=0 \\ \mathcal{L}_0 \triangleright \mathcal{L}_{n-1}(t,x_0,x,x_0',x') & \text{otherwise.}\end{cases}$$

If we abbreviate $\int_{\mathbb{T}^d} \bar{G}(t,y,x)\mu(dy) \int_{\mathbb{T}^d} \bar{G}(t,y',x')\mu(dy')$ as $J_1(t,0,x,0,x')$ we obtain the identity

$$\mathbb{E}[u(t,x)u(t,x')] = J_1(t,0,x,0,x') + \sum_{n=1}^{\infty} \lambda^{2n}\mathcal{L}_n \triangleright J_1(t,0,x,0,x').$$

We can then apply the methods from [5] to prove existence by bounding the series in $\mathcal{L}_n$. The key step is to recognize that we can replace a double convolution in the defintion of $\mathcal{L}_n$ with a pinned brownian motion density. A pinned brownian motion starting at $x_0$ and ending at $x$ at time $t$ on $\mathbb{T}^d$ has density

$$\bar{G}_{t,x_0,x}(s,z) = \frac{\bar{G}(s,x_0,z)\bar{G}(t-s,z,x)}{\bar{G}(t,x_0,x)}.$$

We can then replace two $\bar{G}$ terms with one $\bar{G}_{t,x_0,x}$ in $\mathcal{L}_1$ as follows,

$$\begin{aligned}\mathcal{L}_1(t,x_0,x,x_0',x') &= \int_0^t ds \int\int_{\mathbb{T}^{2d}} dzdz' \bar{G}(t-s,z,x)\bar{G}(t-s,z',x')\bar{G}(s,x_0,z)\bar{G}(s,x_0',z')f(z-z') \\ &= \bar{G}(t,x_0,x)\bar{G}(t,x_0',x') \int_0^t ds \int\int_{\mathbb{T}^{2d}} dzdz' \bar{G}_{t,x_0,x}(s,z)\bar{G}_{t,x_0',x'}(s,z')f(z-z').\end{aligned}$$

We then control the magnitude of the pinned brownian motion in terms of the heat kernel $\bar{G}$ to bound $\mathcal{L}_n$ with respect to $\mathcal{L}_{n-1}$, and we obtain a convergent geometric series to show existence and uniqueness. Dalang's condition controls contributions from the noise. The density for a pinned Brownian motion is well defined for a general manifold, for instance in [17], so we should be able to apply these methods for a general manifold as well.

## 4. Future Directions

A key feature of stochastic processes in random media – what pAm models – is tall peaks isolated in time that occur at different length scales. The first property is called intermittency, and the second property is called multi-fractality. Mandelbrot studied numerous natural systems displaying these phenomena in [36]. In [25] Khoshnevisan, Kim, and Xiao showed that the latter property is distinct from the former and related it to the Haussdorf dimension of certain level sets defined by a gauge function.

Intermittency is defined by the moment Lyapunov exponents,

$$\gamma(k) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^k].$$

It is possible to show that $\gamma(1) \leq \frac{\gamma(2)}{2} \leq \cdots \frac{\gamma(k)}{k} \leq \cdots$. If the inequalities are strict, then the solution has the intermittent property. Carmona and Molchanov point out in [3] that if any inequality is strict than all are, so we often restrict our attention to $\gamma(2)$. Conus and Khoshnevisan showed in [8] that the intermittent property also holds for the general stochastic heat equation on $\mathbb{R}$ depending on a linear growth condition on the coefficient of the noise. For the pAm this is equivalent to $\lambda$ having a critical value where intermittency sets in. Chen and Kim generalized this result to the stochastic heat equation on $\mathbb{R}^d$ in [5].

We hope to further generalize these results about intermittency to the pAm on a manifold. We expect that existence and uniqueness of the solution will depend solely on Dalang's condition, i.e. the singularity of the noise. We hope to explore the relationship between the critical value for $\lambda$ for the onset of intermittency and $\zeta$ for the presence of Hölder continuity viz-a-viz manifold geometry and topology.

I would like to continue to study fBm and rough paths and eventually use these one dimensional processes as measuring rods to study SPDEs. I believe tools from the world of differential geometry such as fiber bundles and jet spaces can motivate the incorporation of rough paths into a differential geometric perspective on SPDEs, building on the work of Martin Hairer on regularity structures.

A longer term project is to study qualitative properties of the intermittence phenomenon, perhaps using the regularity structure theory in a differential geometric approach to the question. I am also interested in interdisciplinary collaboration in the related discipline of quantum field theory as well as chemistry and biology. I believe it is essential to popularize mathematics and present mathematical results to the non-specialist using tools like computer programming, and I hope to take part in math circles and other youth mentoring opportunities related to math.

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